# Perturbation Methods for Algebraic Equations

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*Abstract*- **Perturbation theory is a large collection of iterative methods for obtaining approximation solutions to problems with small parameters like ε, these problems cannot be solved exactly. In this study, we introduce a small ε temporarily into quadratic and cubic problems to find expansion for their roots, after that the case of higher order algebraic problems is considered.**

***Keywords- perturbation; algebraic equation; asymptotic expansion.***

## I. INTRODUCTION

Perturbation methods are mathematical methods that is used to find an approximate solution to a problem which cannot be solved exactly, by starting from the exact solution of a related problem. Perturbation theory is applicable if the problem at hand can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem.
In this paper we discuss perturbation theory for algebraic equations.
Algebraic equations involving small parameter ε are not so easy to be solved by usual methods; in addition to that, they have no exact solutions. Perturbation theory presents efficient and powerful methods to obtain approximate solutions to such algebraic equations. The main goal of perturbation method is to decompose a rough problem into an infinite number of relatively easy ones. Solutions here are represented as an asymptotic expansion in terms of the small parameter, such expansion are called parameter perturbations. In this study we begin with quadratic equations, since their exact solutions are available for comparison. Next we consider cubic equations and finally higher order equations.

## II. RELIMINARIES

Here we introduce some basic definitions, theorems and notes such as binomial theorem, asymptotic expansions, order symbols, uniformity that we need later in our study.

BINOMIAL THEOREM:
The binomial theorem helps us to expand quantities of any given power without direct multiplication. Using straight multiplication, we have:

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b)^3 = a^3 + 3a^2b + 3a\,b^2 + b^3.$$

The process can be generalized for general n as:

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \cdots \quad (1)$$

Equation (1) terminates when n is positive integer. If it does not terminate, then it is true for any positive or negative number n such that $|b/a|$ is less than 1.

ORDER SYMBOLS:
Here we will discuss two order symbols big-oh O[.] and little-oh o[.].

Big-oh:
We define big-oh as:

$$f(\varepsilon) = O[g(\varepsilon)] \quad \text{as } \varepsilon \to 0$$

If

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = A \quad \text{where} \quad 0 < |A| < \infty$$

Little-oh:
We define little-oh as:

$$f(\varepsilon) = o[g(\varepsilon)] \quad \text{as } \varepsilon \to 0$$

If

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0$$

ASYMPTOTIC EXPANSIONS:

DEFINITION 1: The functions $\varphi_1, \varphi_2, \varphi_3, \ldots$ form an asymptotic sequence, or well ordered, as $\varepsilon \to \varepsilon_0$ if and only if $\varphi_n = o(\varphi_m)$ as$\varepsilon \to \varepsilon_0$ forall $m$ and $n$ that satisfy $m < n$.

DEFINITION 2: If $\varphi_1, \varphi_2, \varphi_3, \ldots$ is an asymptotic sequence, then $f(\varepsilon)$ has an asymptotic expansion to $n$ terms, with respect to this sequence, if and only if

$$f = \sum_{k=1}^{m} a_k \varphi_k(\varepsilon) + o(\varphi_m)$$

for $m = 1, \ldots, n$ as $\varepsilon \to \varepsilon_0$, where the $a_k$'s are independent of $\varepsilon$ In this case, we write (as $\varepsilon \to \varepsilon_0$):

$$f \sim a_1\varphi_1(\varepsilon) + a_2\varphi_2(\varepsilon) + a_3\varphi_3(\varepsilon) + \cdots + a_n\varphi_n(\varepsilon).$$

UNIFORMITY OR BREAKDOWN

Suppose we want to represent $f(x,\varepsilon)$ for $x \epsilon D$, by asymptotic expansion as $f(x,\varepsilon) \sim \sum_{n=1}^{m} a_n(x)\varphi_n(\varepsilon)$, This expansion is uniformity valid if

$$a_N(x)\varphi_N(\varepsilon) = o(a_{N-1}(x)\varphi_{N-1}(\varepsilon))$$

as $\varepsilon \to \varepsilon_0$, for every $N \geq 1$, and every $x \epsilon D$, while the expansion break down (or non-uniform), if there is some $x \epsilon D$, and some $N \geq 1$, such that

$$a_N(x)\varphi_N(\varepsilon) = O\left(a_{N-1}(x)\varphi_{N-1}(\varepsilon)\right).$$

## III. PERTURBATION METHOD FOR ALGEBRAIC EQUATIONS

Here we will discuss the perturbation method for quadratic equations, then cubic equations and finally the n-th order equations

QUADRATIC EQUATIONS:

In quadratic equations we will discuss two examples.

EXAMPLE 1:

Consider the equation

$$x^2 - (3 + 2\varepsilon)x + 2 + \varepsilon = 0. \qquad (2)$$

Equation (2) is called perturbed equation. When we put $\varepsilon = 0$, equation (2) becomes

$$(x-2)(x-1) = 0. \qquad (2)$$

Equation(3)is called unperturbed or reduced equation, and the roots of this equation are $x = 1$ and $x = 2$. When ε is small but not zero, the roots will deviate slightly from 1 and 2.

To determine an approximate solution we assume that the roots have expansion of form

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \qquad (4)$$

Then substitute the assumed expansion (4) into equation (2) to get:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 - [(3 + 2\varepsilon)$$
$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)] + 2 + \varepsilon = 0 \qquad (5)$$

After carrying out some operations:

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 = {x_0}^2 + 2\varepsilon x_0 x_1 + \varepsilon^2[\left(2\varepsilon x_0 x_2 + {x_1}^2\right)] + \cdots \qquad (6)$$

Where only terms up to $O(\varepsilon^2)$ have been retained.

$$(3 + 2\varepsilon)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) = \varepsilon(3x_1 + 2x_0)$$
$$3x_0 + \varepsilon^2(3x_2 + 2x_1) + \cdots \qquad (7)$$

Substituting (6) and (7) into(5)we get:

$$({x_0}^2 - 3x_0 + 2) + \varepsilon(2x_0 x_1 - 3x_1 - 2x_0 + 1) + \varepsilon^2(2x_0 x_2 + {x_1}^2 - 3x_2 - 2x_1) + \cdots = 0 \qquad (8)$$

Next equating the coefficients of each power of $\varepsilon$ to zero, yields:

$${x_0}^2 - 3x_0 + 2 = 0, \qquad (9)$$

$$2x_0 x_1 - 3x_1 - 2x_0 + 1 = 0, \qquad (10)$$

$$2x_0 x_2 + {x_1}^2 - 3x_2 - 2x_1 = 0. \qquad (11)$$

After solving these three equations we get values for $x_0, x_1$, and $x_2$, when substituting these values in equation (4) we have:

$$x = 1 - \varepsilon + 3\varepsilon^2 + \cdots,$$

$$x = 2 + 3\varepsilon - 3\varepsilon^2 + \cdots.$$

Finally the exact solution of equation (2) is:

$$x = \frac{1}{2}[3 + 2\varepsilon \pm \sqrt{(3+2\varepsilon)^2 - 4(2+\varepsilon)}] \qquad (12)$$

By using the binomial theorem, we have:

$$(1 + 8\varepsilon + 4\varepsilon^2)^{1/2} = 1 + 4\varepsilon + 2\varepsilon^2 - \frac{1}{8}(64\varepsilon^2 + \cdots)$$
$$= 1 + 4\varepsilon - 6\varepsilon^2 + \cdots$$

substituting in(12) we get:

$$x = \{ \begin{matrix} 2 + 3\varepsilon - 3\varepsilon^2 + \cdots \\ 1 - \varepsilon + 3\varepsilon^2 + \cdots \end{matrix}$$

Which is in agreement with the solution we get by perturbation method.

EXAMPLE 2:
Consider the equation

$$(x-1)(x-a)=-\varepsilon x \qquad (13)$$

When $\varepsilon = 0$, equation (13) reduces to

$$(x-1)(x-a)=0$$

The roots of the last reduced equation are $x = 1$ and $x = a$. We need approximations to the roots of (13) in the form

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \qquad (14)$$

Substituting (14) into(13), we have:
$[(x_0 - 1 + \varepsilon x_1 + \cdots)(x_0 - a + \varepsilon x_1 + \cdots)] = -\varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)$
Which upon expanding, yields:
$(x_0-1)(x_0-a) + \varepsilon(x_0-1)x_1 + \varepsilon^2(x_0-1)x_2 + \varepsilon(x_0-a)x_1 + \varepsilon^2 {x_1}^2 + \varepsilon^2(x_0-a)x_2 + \varepsilon x_0 + \varepsilon^2 x_1 + \cdots = 0$
Collecting coefficients of like power of ε gives:

$$(x_0-1)(x_0-a) + \varepsilon[(2x_0 - 1 - a)x_1 + x_0] + \varepsilon^2[(2x_0 - 1 - a)x_2 + {x_1}^2 + x_1] + \cdots = 0 \qquad (15)$$

It is clear that only terms up to $O(\varepsilon^2)$ have been retained. Equating the coefficient of each power of ε in (15) to zero, we get:

$$(x_0-1)(x_0-a) = 0, \qquad (16)$$

$$(2x_0 - 1 - a)x_1 + x_0 = 0, \qquad (17)$$

$$[(2x_0 - 1 - a)x_2 + {x_1}^2 + x_1] = 0. \qquad (18)$$

The solutions of (16) are:
$x_0 = 1$, or $x_0 = a$
When $x_0 = 1$, equation (17) becomes:

$$(1-a)x_1 + 1 = 0, \text{ so that } x_1 = -\frac{1}{1-a}$$

Then (18) becomes:

$$(1-a)x_2 = -\frac{1}{(1-a)^2} + \frac{1}{1-a} = -\frac{a}{(1-a)^2}$$

or

$$x_2 = -\frac{a}{(1-a)^3}$$

Hence, the first root is:

$$x = 1 - \frac{\varepsilon}{1-a} - \frac{a\varepsilon^2}{(1-a)^3} + \cdots \qquad (19)$$

When $x_0 = a$, equation (17) becomes:

$$(a-1)x_1 + a = 0 \text{ so that } x_1 = \frac{a}{1-a}$$

Then, (18) becomes:

$$(a-1)x_2 = -\frac{a}{1-a} - \frac{a^2}{(1-a)^2} = -\frac{a}{(1-a)^2}$$

or

$$x_2 = \frac{a}{(1-a)^3}$$

Hence, the second root is:

$$x = a + \frac{a\varepsilon}{1-a} + \frac{a\varepsilon^2}{(1-a)^3} + \cdots \qquad (20)$$

Equations (19) and (20) indicate that our expansion breakdown when $a \to 1$, $a$ need not be exactly equal 1 for the above expansion to breakdown. According to the definition of non-uniformity we will determine the condition under which successive terms are the same order, it follows from (19) that the zeroth order and the first order term have the same order when

$$\frac{\varepsilon}{1-a} = O(1) \text{ or } 1-a = O(\varepsilon)$$

Whereas the first order and second order terms are the same order when

$$\frac{\varepsilon}{1-a} \quad O\left(\frac{\varepsilon^2}{(1-a)^3}\right) \quad \text{or} \quad (1-a)^2 = O(\varepsilon)$$

Since for small ε, $\varepsilon^{1/2}$ is bigger than ε, the region of nonuniformity is $1 - a = O\left(\varepsilon^{1/2}\right)$, the larger of the two region.
Now to investigate the exact solution. We rewrite (13) as:

$$x^2 - x - ax + a + \varepsilon x = 0$$

or

$$x^2 - (1 + a - \varepsilon)x + a = 0$$

Whose roots are given by:

$$x = \frac{1 + a - \varepsilon \pm \sqrt{(1+a-\varepsilon)^2 - 4a}}{2} = \frac{1 + a - \varepsilon \pm \sqrt{(1-a)^2 - 2\varepsilon(1+a) + (\varepsilon)^2}}{2} \qquad (21)$$

Next, we expand (21) for small ε and compare the result with (19) and (20) using the binomial theorem, we have:

$$[(1-a)^2 - 2\varepsilon(1+a) + (\varepsilon)^2]^{1/2} = (1-a)[1 - \frac{\varepsilon(1+a)}{(1-a)^2} + \frac{(\varepsilon)^2}{2(1-a)^2} - \frac{4\varepsilon^2\ (1+a)^2}{8(1-a)^4} + \cdots]$$

$$= (1-a)\left[1 - \frac{\varepsilon(1+a)}{(1-a)^2} - \frac{2\varepsilon^2\ a}{(1-a)^4} + \cdots\right] \qquad (22)$$

And also here the terms have been retained up to $O(\varepsilon^2)$.
Putting (22) into (21) with the positive sign gives one of the roots as:

$$x = \frac{1}{2}[1 + a - \varepsilon + 1 - a - \frac{\varepsilon(1+a)}{1-a} - \frac{2\varepsilon^2\ a}{(1-a)^3} + \cdots] \qquad (23)$$

or

$$x = 1 - \frac{\varepsilon}{1-a} - \frac{\varepsilon^2\ a}{(1-a)^3} + \cdots$$

Now putting (22) into (21) with the negative sign gives the second root as:

$$x = \frac{1}{2}[1 + a - \varepsilon - 1 + a + \frac{\varepsilon(1+a)}{1-a} + \frac{2\varepsilon^2\ a}{(1-a)^3} + \cdots]$$

or

$$x = a + \frac{\varepsilon a}{1-a} + \frac{\varepsilon^2\ a}{(1-a)^3} + \cdots \qquad (24)$$

In arriving at (23) and (24) from the exact solution. The subtraction and addition operation are usually justified, so, the exponentiation operation is the suspect operation. In approximating

$$(1-u)^{1/2} \text{ by } 1-\frac{u}{2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)u^2}{2!}+\cdots$$

We made the implicit assumption that $|u| < 1$. in the present example,

$$u = \frac{2\varepsilon(1+a)^2-\varepsilon^2}{(1-a)^2} \quad (25)$$

And this magnitude is small when we compare it with 1 only when $a$ is away from 1. At $a = 1$, $u = \infty$, irrespective of small ε is as long as it is from zero. (25) show that the binomial expansion in not justified when $u = O(1)$ or $(1-a)^2 = O(\varepsilon)$ or $1-a = O(\varepsilon^{1/2})$

So, to obtain a uniform expansion when $1-a = O(\varepsilon^{1/2})$, we must modify above procedure by taking this fact into our account. this can be done by introducing the parameter σ we defined it by:

$$1-a = \varepsilon^{1/2}\sigma \quad (26)$$

Where σ is independent of ε. Putting (26) in (13) gives:

$$(x-1)\left(x-1+\varepsilon^{1/2}\sigma\right) = -\varepsilon x \quad (27)$$

When $\varepsilon = 0$, (27) reduces to:

$$(x-1)^2 = 0$$

Which yields the double root $x = 1$. This fact and the presence of $\varepsilon^{1/2}$ in (27) suggest trying an expansion in the form:

$$x = 1+\varepsilon^{1/2}x_1+\cdots \quad (28)$$

We stop at $O(\varepsilon^{1/2})$ because obtaining the higer-order terms is straightforward substituting the first two terms in (28) into (27) gives;

$$\left(\varepsilon^{1/2}x_1+\cdots\right)\left(\varepsilon^{1/2}x_1+\varepsilon^{1/2}\sigma+\cdots\right) = -\varepsilon(1+\varepsilon^{1/2}x_1+\cdots)$$

or

$$\varepsilon x_1^2+\varepsilon\sigma x_1+\varepsilon+\cdots = 0$$

By collecting the coefficient of the same power to zero we get:

$$x_1^2+\sigma x_1+1 = 0$$

Whose roots are:

$$x_1 = \frac{(-\sigma)\pm\sqrt{\sigma^2-4}}{2}$$

Therefore, the roots of (13) in this case are given by:

$$x = 1-\varepsilon^{1/2}\left(\frac{(\sigma)+\sqrt{\sigma^2-4}}{2}\right)+\cdots$$

$$x = 1-\varepsilon^{1/2}\left(\frac{(\sigma)-\sqrt{\sigma^2-4}}{2}\right)+\cdots$$

CUBIC EQUATIONS:

Here we will view two examples:

EXAMPLE 1:

Consider the equation

$$x^3-(6+\varepsilon)x^2+(11+2\varepsilon)x-6+\varepsilon^2 = 0 \quad (29)$$

We assume that the roots have expansion in the form:

$$x = x_0+\varepsilon x_1+\cdots \quad (30)$$

Substituting (30) into (29)to get:

$$x_0^3+3\varepsilon x_0^2x_1-6x_0^2-12\varepsilon x_0x_1-\varepsilon x_0^2+11x_0+11\varepsilon x_1+2\varepsilon x_0-6+\cdots$$

Collecting coefficients of equal powers of ε gives:

$$x_0^3-6x_0^2+11x_0-6+\varepsilon\,(3x_0^2x_1-12x_0x_1+11x_1-x_0^2+2x_0)+\cdots = 0$$

Where terms up to $O(\varepsilon)$ have been retained, consistent with our expansion. Equating the coefficients of the same power ε of to zero yields:

$$x_0^3-6x_0^2+11x_0-6 = 0 \quad (31)$$

$$3x_0^2x_1-12x_0x_1+11x_1-x_0^2+2x_0 = 0 \quad (32)$$

By solving equation (31) we get that $x_0 = 1$, $x_0 = 2$, $x_0 = 3$. It follows that:

$$x_1 = \frac{x_0^2-2x_0}{3x_0^2-12x_0+11} \quad (33)$$

When $x_0 = 1$, it follows from (33) that $x_1 = \frac{-1}{2}$. So the expansion given by: $x = 1-\frac{1}{2}\varepsilon+\cdots$

When $x_0 = 3$, it follows from(33) that $x_1 = \frac{3}{2}$. Hence, the expansion given by: $x = 3+\frac{3}{2}\varepsilon+\cdots$

And when $x_0 = 2$, it follows from (33) that $x_1 = 0$, Hence, the expansion given by: $x = 2+(0)\varepsilon+\cdots$.

EXAMPLE 2:

Consider the equation

$$\varepsilon x^3+x+2+\varepsilon = 0 \quad (34)$$

As $\varepsilon \to 0$, (34) reduces to:

$$x+2 = 0$$

Again, let us try an expansion in the form

$$x = x_0+\varepsilon x_1+\cdots \quad (35)$$

Substituting (35) into (34) we get:

$$\varepsilon(x_0+\varepsilon x_1+\cdots)^3+x_0+\varepsilon x_1+\cdots+2+\varepsilon = 0$$

or

$$x_0+2+\varepsilon(x_1+x_0^3+1)+\cdots = 0$$

Equating coefficients of same powers of ε, we have:

$$x_0+2 = 0$$

$$x_1+x_0^3+1 = 0$$

Hence, $x_0 = -2$ and $x_1 = 7$. So, one of the roots is given by:

$$x = -2+7\varepsilon+\cdots$$

We note that the other roots tend to ∞ as $\varepsilon \to 0$, to find these roots, we introduce a variable y, where

$$x = \frac{y}{\varepsilon^v}\quad, v > 0 \quad (36)$$

Substituting (36) into (34), we have

$$\varepsilon^{1-3v}y^3+\varepsilon^{-v}y+2+\varepsilon = 0 \quad (37)$$

In order to obtain a nontrivial solutions, we require that at least two leading-order terms in equation (37) have the same order of magnitude. This is called the principle of dominant balance.

In equation (37) balancing the first two terms, we find that:

$$\varepsilon^{2v} = \varepsilon$$

$$2v = 1 \ \ or \ \ \ v = \frac{1}{2}$$

By substituting the value of $v$ in equation (37) we have

$$\varepsilon^{-1/2}\ y^3 + \varepsilon^{-1/2} y + 2 + \varepsilon = 0$$

or

$$y^3 + y + 2\,\varepsilon^{1/2} + \varepsilon^{3/2} = 0 \qquad (38)$$

For equation (38) suppose another expansion, so let

$$y = y_0 + y_1\,\varepsilon^{1/2} + \cdots \qquad (39)$$

Now substitute(39) into (38) we get:

$$(y_0 + y_1\,\varepsilon^{1/2} + \cdots)^3 + (y_0 + y_1\,\varepsilon^{1/2} + \cdots) + 2\,\varepsilon^{1/2} + \cdots = 0$$

or

$$y_0^3 + 3y_0^2 y_1\,\varepsilon^{1/2} + y_0 + y_1\,\varepsilon^{1/2} + 2\,\varepsilon^{1/2} + \ldots = 0$$

Now equating coefficients of the same power of ε, we have:

$$y_0^3 + y_0 = 0 \qquad (40)$$

$$3y_0^2 y_1 + y_1 + 2 = 0 \qquad (41)$$

The solutions of (40) are:

$y_0 = 0$, $y_0 = i$, and $y_0 = -i$.

When $y_0 = 0$, equation (41) becomes

$y_1 + 2 = 0$. So that $y_1 = -2$.

Hence, the first root is:

$$y = 0 - 2\,\varepsilon^{1/2} + \cdots$$

The corresponding solutions for $x$ are

$x = -2 + \cdots$, we notice that this root corresponds to the first root, so we discarded it.

When $y_0 = \pm i$, (41) becomes:

$-2y_1 + 2 = 0$ so that $y_1 = 1$.

So the second and the third root are

$$y = \pm i + \varepsilon^{1/2} + \cdots$$

The corresponding solutions for x are

$$x = \frac{\pm i}{\varepsilon^{1/2}} + 1 + \cdots.$$

HIGHER ORDER EQUATIONS:

Now, we will talk about higher–order equations, and we will focus on the case that the small parameter multiplies the highest power of $x$. We have:

$$\varepsilon x^n = x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \qquad (42)$$

$a's$ are constants independent of ε and $x$, n and m are intgers, and n > m. when $\varepsilon \to 0$, equation(42) becames:

$$x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 = 0 \qquad (43)$$

This equation has $\alpha_s$ roots, where $s = 1,2,\ldots,m$.

Let the expansion of the root as:

$$x = x_0 + \varepsilon x_1 + \cdots \qquad (44)$$

Substitute (44) in (42), we have:

$$\varepsilon\left(x_0 + \varepsilon x_1 + \cdots\right)^n = \left(x_0 + \varepsilon x_1 + \cdots\right)^m + a_{m-1}[\left(x_0 + \varepsilon x_1 + \cdots\right)^{m-1}] + a_{m-2}\left(x_0 + \varepsilon x_1 + \cdots\right)^{m-2} + \cdots + a_1\left(x_0 + \varepsilon x_1 + \cdots\right) + a_0$$

or

$$x_0{}^m + a_{m-1}x_0{}^{m-1} + a_{m-2}x_0{}^{m-2} + \cdots + a_1 x_0 + a_0 + \varepsilon[m x_0{}^{m-1} + (m-1)a_{m-1}x_0{}^{m-2} + (m-2)a_{m-2}x_0{}^{m-3} + \cdots + a_1]x_1 - \varepsilon x_0{}^n + O(\varepsilon^2) = 0$$

Equating the coefficients of like powers of $\varepsilon$ we have,

$$x_0{}^m + a_{m-1}x_0{}^{m-1} + \cdots + a_1 x_0 + a_0 = 0 \qquad (45)$$

Equation (45) has the roots $x_0 = \alpha_s$, where $s = 1,2,\ldots,m$, it is follows that:

$$x_1 = \alpha_s{}^n[m\alpha_s{}^{m-1} + (m-1)a_{m-1}\alpha_s{}^{m-2} + (m-2)a_{m-2}\alpha_s{}^{m-3} + \cdots + a_1]^{-1}$$

Hence,

$$x = \alpha_s + \varepsilon\alpha_s{}^n[m\alpha_s{}^{m-1} + (m-1)a_{m-1}\alpha_s{}^{m-2} + (m-2)a_{m-2}\alpha_s{}^{m-3} + \cdots + a_1]^{-1} + \cdots \qquad (46)$$

(46) Breaks down when the term inside brackets go to zero, in this case, the expansion goes in fractional power of $\varepsilon$ and we need to follow the procedure used in example 2 in quadratic equation.

We have $(n-m)$ other roots, which tend to ∞ as $\varepsilon \to 0$ because $\varepsilon$ multiples the highest power of $x$. To find these roots we introduce a variable y, where

$$x = \frac{y}{\varepsilon^v}, \quad v > 0 \qquad (47)$$

Substituting (47) into(42), we have

$$-\varepsilon\left(\frac{y}{\varepsilon^v}\right)^n + \left(\frac{y}{\varepsilon^v}\right)^m + a_{m-1}\left(\frac{y}{\varepsilon^v}\right)^{m-1} + a_{m-2}\left(\frac{y}{\varepsilon^v}\right)^{m-2} + \ldots + a_1\frac{y}{\varepsilon^v} + a_0 = 0 \qquad (48)$$

In equation (48 ) balancing the first two terms, we find that

$$\varepsilon^{(1-nv)} = \varepsilon^{-mv} \ \text{ or } \ \varepsilon^{v(n-m)} = \varepsilon$$

$$v(n-m) = 1 \ \text{ or } \ v = \frac{1}{n-m}$$

If we multiply all terms in equation(48) by $\varepsilon^{(nv-1)}$ we get:

$$y^n = y^m + a_{m-1}\,y^{m-1}\varepsilon^v + \cdots + a_1 y\,\varepsilon^{v(n-1)-1} + a_0\varepsilon^{nv-1} \qquad (49)$$

For equation (49) suppose another expansion as follows:

$$y = (y_0 + y_1\,\varepsilon^v + \cdots) \qquad (50)$$

Now substituting(50) into (49) we get:

$$(y_0 + y_1\,\varepsilon^v + \cdots)^n = (y_0 + y_1\,\varepsilon^v + \cdots)^m + a_{m-1}(y_0 + y_1\,\varepsilon^v + \cdots)^{m-1}\varepsilon^v + \cdots + a_1(y_0 + y_1\,\varepsilon^v + \cdots)\,\varepsilon^{v(n-1)-1} + a_0\varepsilon^{v(n-1)}$$

Equating coefficients of same powers of $\varepsilon$, we have

$$y_0{}^m = y_0{}^n \ \text{ or } \ y_0{}^{n-m} = 1 = e^{2ir\pi}$$

Where $r = 1,2,3,\ldots,(n-m)$.

Hence,

$$y_0 = \omega, \omega^2, \ldots, \omega^k, \quad \omega = exp\,(2i\pi/(n-m)), \ k = (n-m)$$

Now equating the coefficients of $\varepsilon^v$ to get

$$n \ \ y_0{}^{n-1} y_1 = m \ \ y_0{}^{m-1} y_1 + \ a_{m-1} \ \ y_0{}^{m-1}$$

We get

$$y_1 = \frac{a_{m-1} y_0{}^{m-1}}{n y_0{}^{n-1} - m y_0{}^{m-1}} = \frac{a_{m-1}}{n y_0{}^{n-m} - m} = \frac{a_{m-1}}{n-m}$$

Now substitute the root in equation $(50)$, so we have:

$$y = \omega^r + \frac{a_{m-1}}{n-m} \ \varepsilon^v + \cdots$$

The corresponding solutions for x are

$$x = \frac{\omega^r}{\varepsilon^v} + \frac{a_{m-1}}{n-m} + \cdots$$

Where $r = 1,2,3, \ldots, n-m$.

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